Some a priori estimate related to the well-posedness for the barotropic compressible Navier-Stokes system (Harmonic Analysis and Nonlinear Partial Differential Equations)

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B65 (2017), 001–022

Some a priori estimate related to the well-posedness for the barotropic compressible Navier-Stokes system

By

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Abstract

We consider some global a priori estimate related to the well-posedness for the barotropic compressible Navier-Stokes system. We employ the Fourier-Herz spaces instead of the standard Besov spaces, and show a global in time a priori bound of the solutions for the initial data in the L^q-type regularity framework.

§ 1. Introduction : barotropic compressible Navier-Stokes system

We consider the following barotropic compressible Navier-Stokes system:

(1.1)
$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + (\mu + \lambda)\nabla \mathrm{div}\, u, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), & x \in \mathbb{R}^d. \end{cases}$$

The unknowns are $\rho = \rho(t,x) \in \mathbb{R}_+$ and $u = u(t,x) \in \mathbb{R}^d$ representing the fluid density and the velocity vector field, respectively. We assume that there is no vacuum, and ρ tends to a positive constant $\bar{\rho}$ at spatial infinity. The viscosity coefficients λ and μ are given constants satisfying $\mu > 0$ and $\lambda + 2\mu > 0$. We assume that the pressure is given by $P := P(\rho)$, where P is a sufficiently smooth function and that $P'(\bar{\rho}) > 0$.

§ 1.1. Some notations and the up-to-date critical theory

Hereafter, we denote by $L^p(\mathbb{R}^d) = L^p$ $(1 \le p \le \infty)$ standard Lebesgue spaces on $\mathbb{R}^d$, and by ℓ^p the set of sequences with summable p-th powers. The space of all Schwarz

Received October 26, 2016. Revised January 10, 2017.
2010 Mathematics Subject Classification(s): 76N10, 35Q30
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functions is denoted by $\mathcal{S}(\mathbb{R}^d)$ and the Fourier transform of a function (or a tempered distribution) u is denoted by $\widehat{u}$ or $\mathcal{F}u$. Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be a Littlewood-Paley dyadic decomposition of unity. Namely, let $\widehat{\phi} \in \mathcal{S}(\mathbb{R}^d)$ be a non-negative radially symmetric function that satisfies

$$\operatorname{supp}\widehat{\phi} \subset \{\xi \in \mathbb{R}^n; 2^{-1} < |\xi| < 2\},$$
$$\widehat{\phi_j}(\xi) := \widehat{\phi}(2^{-j}\xi) \text{ (for all } j \in \mathbb{Z}) \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \widehat{\phi_j}(\xi) = 1 \text{ (for all } \xi \neq 0).$$

We further set $\widehat{\Phi} := 1 - \sum_{j\geq 1} \widehat{\phi_j}$ and $\dot{S}_m u := \Phi(2^{-m}\cdot) * u$, for $m \in \mathbb{Z}$.

Definition 1.1 (Homogeneous Besov spaces). *Let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions on $\mathbb{R}^d$. For $1 \leq p \leq \infty$ and $s \leq n/p$, we denote by $\dot{B}^s_{p,1}(\mathbb{R}^d)$ (or more simply $\dot{B}^s_{p,1}$) the space of tempered distributions u so that*

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \quad \textit{in} \quad \mathcal{S}'(\mathbb{R}^n) \quad \textit{with} \quad \dot{\Delta}_j u := \phi_j * u$$

and

$$\|u\|_{\dot{B}^s_{p,1}} := \sum_{j\in\mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L^p} < \infty.$$

Having fixed some $k_0 \in \mathbb{Z}$, we denote by $u_L := \dot{S}_{k_0} u$ the low frequencies of u, and by $u_H := u - u_L$ the high frequencies of u. We shall also need the notation

$$\|u\|^L_{\dot{B}^s_{p,1}} := \sum_{j\leq k_0} 2^{js}\|\dot{\Delta}_j u\|_{L^p} \quad \text{and} \quad \|u\|^H_{\dot{B}^\sigma_{p,1}} := \sum_{j\geq k_0-1} 2^{j\sigma}\|\dot{\Delta}_j u\|_{L^p}. \tag{1.2}$$

Note the small overlap between low and high frequencies, ensuring that

$$\|u_L\|_{\dot{B}^s_{p,1}} \leq C\|u\|^L_{\dot{B}^s_{p,1}} \quad \text{and} \quad \|u_H\|_{\dot{B}^\sigma_{p,1}} \leq C\|u\|^H_{\dot{B}^\sigma_{p,1}}.$$

Given some Banach space X, we denote by $L^q(0,T;X)$ the corresponding Bochner space and write its norm $\|\cdot\|_{L^q_T(X)}$. We denote the fractional derivative by $\Lambda^s := \mathcal{F}^{-1}|\xi|^s\mathcal{F}$ for $s \in \mathbb{R}$. We set $a := \rho - \bar{\rho}$ and define

$$\begin{aligned} X_p(t) :=& \|a\|^L_{\widetilde{L^\infty_t}(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|\Lambda^2 a\|^L_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|u\|^L_{\widetilde{L^\infty_t}(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|\Lambda^2 u\|^L_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\ &+ \|a\|^H_{\widetilde{L^\infty_t}(\dot{B}^{\frac{d}{p}}_{p,1})} + \|a\|^H_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|u\|^H_{\widetilde{L^\infty_t}(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\Lambda^2 u\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}. \end{aligned} \tag{1.3}$$

Here, the tilde norms are defined by

$$\|u\|_{\widetilde{L^r_t}(\dot{B}^\sigma_{p,1})} := \sum_{j\in\mathbb{Z}} 2^{j\sigma}\|\dot{\Delta}_j u\|_{L^r(0,t;L^p)}. \tag{1.4}$$

The global critical regularity theory for the compressible viscous fluid initiated by Danchin [6] gave rise to many global existence results [5, 3, 7, 9]. The following is one of such global existence results.

Theorem 1.2 ([5, 7]). *Let $d \geq 2$ and $p \in [2, \min(4, \frac{2d}{d-2})]$ with, additionally, $p \neq 4$ if $d = 2$. Assume with no loss of generality that $P'(1) = 1$ and $\nu = 1$. There exists a universal integer $k_0 \in \mathbb{N}$ and a small constant $c = c(p, d, \mu, P)$ such that if $a_0 := \rho_0 - \bar\rho \in \dot B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ and $u_0 \in \dot B^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d)$ with besides $(a_0, u_0)_L$ in $\dot B^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$ (with the notation $z_L := \dot S_{k_0} z$ and $z_H := z - z_L$) satisfy*

$$X_{p,0} := \|(a_0, u_0)\|^L_{\dot B^{\frac{d}{2}-1}_{2,1}} + \|a_0\|^H_{\dot B^{\frac{d}{p}}_{p,1}} + \|u_0\|^H_{\dot B^{\frac{d}{p}-1}_{p,1}} \leq c$$

then (1.1) has a unique global-in-time solution $(a, u) := (\rho - \bar\rho, u)$ in the space X_p defined by

$$(a, u)_L \in \widetilde C(\mathbb{R}_+; \dot B^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)) \cap L^1(\mathbb{R}_+; \dot B^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d)),$$
$$a_H \in \widetilde C(\mathbb{R}_+; \dot B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)) \cap L^1(\mathbb{R}_+; \dot B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)),$$
$$u_H \in \widetilde C(\mathbb{R}_+; \dot B^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d)) \cap L^1(\mathbb{R}_+; \dot B^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d))$$

where we denoted $\widetilde C(\mathbb{R}_+; \dot B^s_{q,1}(\mathbb{R}^d)) := C(\mathbb{R}_+; \dot B^s_{q,1}(\mathbb{R}^d)) \cap \widetilde L^\infty(\mathbb{R}_+; \dot B^s_{q,1}(\mathbb{R}^d))$. Furthermore, we have for some constant $C = C(p, d, \mu, P)$,

$$X_p(t) \leq C X_{p,0} \quad \textit{for all} \quad t \geq 0.$$

Note that the restriction on p is rather strong in Theorem 1.2, which amounts to

$$p \in \begin{cases} [2,4) & \text{if} \quad d = 2 \\ [2,4] & \text{if} \quad d = 3 \\ \left[2, \dfrac{2d}{d-2}\right] & \text{if} \quad d \geq 4. \end{cases} \tag{1.5}$$

In particular, p cannot exceed 4 in any dimension due to the following global a priori bound, upon which the proof of Theorem 1.2 essentially depends (see [5, 7]).

Lemma 1.3 ([5, 7]). *Let $d \geq 2$ and $p \in [2, \min(4, \frac{2d}{d-2})]$. Let (a, u) be a solution to System (1.1) in X_p with initial data $a_0 \in \dot B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ and $u_0 \in \dot B^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d)$ with in addition $a_{0L} \in \dot B^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$ and $u_{0L} \in \dot B^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$. Then there exists a universal threshold number $k_0 \in \mathbb{Z}$ and some constant C depending on d, p, μ, λ, P such that if X_p in (1.3) is defined according to k_0 then we have*

$$X_p(t) \leq C(X_{p,0} + X_p^2(t)) \quad \textit{for all} \quad t \geq 0. \tag{1.6}$$

Once one obtains an a priori bound of the above type, it rests upon a standard continuity argument to yield the global bound $X_p(t) \leq CX_{p,0}$ provided $X_{p,0}$ is small enough. To complete this task, one needs the continuity of $X_p(t)$ and the assumption that the local solution exists in the space X_p.

In this paper, we consider the above global a priori estimate (1.6) in the context of the Fourier-Herz spaces defined below. We generalize the assumption on the low frequencies $(a_0, u_0)_L$ of the initial data to the L^q-type regularity framework and establish a global apriori estimate.

§ 1.2. Main result : Global a priori estimate in Fourier-Herz spaces

To state our result, we introduce spaces called Fourier-Herz spaces, which have been used for studies of well-posedness of the incompressible viscous fluids. Lei and Lin [12] considered the three dimensional incompressible Navier-Stokes system in a scale critical space $\mathcal{X}^{-1}(\mathbb{R}^3)$ that is a subspace of $BMO^{-1}(\mathbb{R}^d)$ but does not have inclusion relation with $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Cannone and Wu [2] generalized the result in [12] to slightly larger spaces. Iwabuchi and Takada [11] considered the incompressible rotating Navier-Stokes system and obtained a uniform global solution.

Definition 1.4 (Fourier-Herz spaces). *For $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq \sigma \leq \infty$, we define the Fourier-Herz space $\dot{\mathcal{B}}^s_{p,\sigma}(\mathbb{R}^d)$ as follows:*

$$\dot{\mathcal{B}}^s_{p,\sigma}(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d)\ ; \widehat{u} \in L^1_{loc}(\mathbb{R}^d),\ \|u\|_{\dot{\mathcal{B}}^s_{p,\sigma}} < \infty\},$$

$$\|u\|_{\dot{\mathcal{B}}^s_{p,\sigma}} := \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{js\sigma} \|\widehat{\phi}_j \widehat{u}\|^\sigma_{L^{p'}}\Big)^{\frac{1}{\sigma}}, & 1 \leq \sigma < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\widehat{\phi}_j \widehat{u}\|_{L^{p'}}, & \sigma = \infty, \end{cases}$$

where p' is the Hölder conjugate of p.

Let us be reminded that the following canonical relations hold thanks to Hausdorff-Young inequality :

$$\begin{aligned} \dot{\mathcal{B}}^s_{p,\sigma}(\mathbb{R}^d) &\hookrightarrow \dot{B}^s_{p,\sigma}(\mathbb{R}^d) \quad \text{if} \quad 2 \leq p \leq \infty \\ \text{and} \quad \dot{B}^s_{p,\sigma}(\mathbb{R}^d) &\hookrightarrow \dot{\mathcal{B}}^s_{p,\sigma}(\mathbb{R}^d) \quad \text{if} \quad 1 \leq p \leq 2 \end{aligned}$$

for all $s \in \mathbb{R}$ and $1 \leq \sigma \leq \infty$. Obviously, it holds that $\dot{B}^s_{2,\sigma}(\mathbb{R}^d) = \dot{\mathcal{B}}^s_{2,\sigma}(\mathbb{R}^d)$.

We use similar notations to (1.2):

$$\|u\|^L_{\dot{\mathcal{B}}^s_{p,1}} := \sum_{j \leq k_0} 2^{js} \|\widehat{\phi}_j \widehat{u}\|_{L^{p'}} \quad \text{and} \quad \|u\|^H_{\dot{\mathcal{B}}^\sigma_{p,1}} := \sum_{j \geq k_0 - 1} 2^{j\sigma} \|\widehat{\phi}_j \widehat{u}\|_{L^{p'}}$$

for some threshold integer k_0 dividing the low and high frequencies and define

$$\begin{aligned}X_{q,p}^{k_0}(t) :=& \|a\|^L_{\widetilde{L_t^\infty}(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} + \|\Lambda^2 a\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} + \|u\|^L_{\widetilde{L_t^\infty}(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} + \|\Lambda^2 u\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})}\\ &+ \|a\|^H_{\widetilde{L_t^\infty}(\dot{B}^{\frac{d}{p}}_{p,1})} + \|a\|^H_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|u\|^H_{\widetilde{L_t^\infty}(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\Lambda^2 u\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}.\end{aligned} \tag{1.7}$$

For notational simplicity, we write $X(t) = X_{q,p}^{k_0}(t)$ in what follows. Our main result reads as follows:

Theorem 1.5. *Let $d \geq 2$, $1 \leq q \leq 2$ and $p \in [2, \min(2q, \frac{qd}{d-q})]$. Let (a,u) be a solution to System (1.1) in the space X defined by*

$$\begin{aligned}(a,u)_L &\in \widetilde{C}(\mathbb{R}_+; \dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1}(\mathbb{R}^d)) \cap L^1(\mathbb{R}_+; \dot{\mathcal{B}}^{\frac{d}{q}+1}_{q,1}(\mathbb{R}^d)),\\ a_H &\in \widetilde{C}(\mathbb{R}_+; \dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)),\\ u_H &\in \widetilde{C}(\mathbb{R}_+; \dot{B}^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d)) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d))\end{aligned}$$

where we denoted $\widetilde{C}(\mathbb{R}_+; \dot{B}^s_{q,1}(\mathbb{R}^d)) := C(\mathbb{R}_+; \dot{B}^s_{q,1}(\mathbb{R}^d)) \cap \widetilde{L}^\infty(\mathbb{R}_+; \dot{B}^s_{q,1}(\mathbb{R}^d))$ and likewise for the Fourier-Herz spaces. Assume that initial data satisfies $a_0 \in \dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ and $u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d)$ with in addition $a_{0L} \in \dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1}(\mathbb{R}^d)$ and $u_{0L} \in \dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1}(\mathbb{R}^d)$. Then there exists a universal threshold number $k_0 \in \mathbb{Z}$ and some constant C depending on d, p, μ, λ, P such that if $X(t)$ in (1.7) is defined according to k_0 then we have

$$X(t) \leq C(X_0 + X^2(t)) \quad \textit{for all} \quad t \geq 0,$$

where

$$X_0 := \|(a_0, u_0)\|^L_{\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1}} + \|a_0\|^H_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|u_0\|^H_{\dot{B}^{\frac{d}{p}-1}_{p,1}}.$$

Remark. Theorem 1.5 can be regarded as a generalization of the low frequency assumptions in Lemma 1.3. In fact, taking $q = 2$ recovers Lemma 1.3. However, it does not improve the restriction on p in Lemma 1.3 and the optimal case is achieved when $q = 2$.

Remark. A natural question is whether it is possible to impose critical Fourier-Herz regularity to all frequencies and obtain a well-posedness result without any restriction on p. This is a work in progress.

§ 1.3. Reformulation of the problem

Here, we normalize some coefficients of our problem to simplify the later calculations. Denoting $a := \rho - \bar\rho$, the system (1.1) is linearized around $(\bar\rho, 0)$ as follows:

$$\begin{cases} \partial_t a + \bar\rho \operatorname{div} u = -\operatorname{div}(au), \\ \partial_t u - \dfrac{1}{\bar\rho}\mathcal{L}u + \dfrac{P'(\bar\rho)}{\bar\rho}\nabla a = -u\cdot\nabla u + (\dfrac{P'(\bar\rho)}{\bar\rho} - \dfrac{P'(\rho)}{\rho})\nabla a - (\dfrac{1}{\bar\rho} - \dfrac{1}{\rho})\mathcal{L}u, \\ (a,u)|_{t=0} = (a_0, u_0), \end{cases}$$

where $\mathcal{L}u := \mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u$. Let us split u into compressible and incompressible components, namely, $v := \Lambda^{-1}\operatorname{div} u$ and $w := \Lambda^{-1}\operatorname{rot}\, u$, respectively (recall $\Lambda^s := (-\Delta)^{s/2}$ for $s \in \mathbb{R}$). Then the linearized system for (a, v, w) is

$$\begin{cases} \partial_t a + \bar\rho\Lambda v = 0, \\ \partial_t v - \dfrac{\nu}{\bar\rho}\Delta v - \dfrac{P'(\bar\rho)}{\bar\rho}\Lambda a = 0, \\ \partial_t w - \dfrac{\mu}{\bar\rho}\Delta w = 0, \end{cases}$$

where $\nu := 2\mu + \lambda$. We normalize the coefficients appearing in the first and second equations, that is, we assume that $\bar\rho = P'(\bar\rho) = \nu = 1$, which is justified by the change of variable

$$a(t,x) = \bar\rho\widetilde{a}\left(\frac{\bar\rho c^2}{\nu}t, \frac{\bar\rho c}{\nu}x\right) \quad \text{and} \quad u(t,x) = c\,\widetilde{u}\left(\frac{\bar\rho c^2}{\nu}t, \frac{\bar\rho c}{\nu}x\right),$$

where $c := \sqrt{P'(\bar\rho)}$. Note that now ρ can be expressed as

$$\rho = a + \bar\rho = \bar\rho(\widetilde{a} + 1).$$

Indeed, the above rescaling ensures that $(\widetilde{a}, \widetilde{u})$ satisfies

$$\begin{cases} \partial_t \widetilde{a} + \operatorname{div}\widetilde{u} = -\operatorname{div}(\widetilde{a}\widetilde{u}), \\ \partial_t \widetilde{u} - \dfrac{1}{\nu}\mathcal{L}\widetilde{u} + \nabla\widetilde{a} = -\widetilde{u}\cdot\nabla\widetilde{u} + (1 - \dfrac{P'(\bar\rho(\widetilde{a}+1))}{c^2(\widetilde{a}+1)})\nabla\widetilde{a} - \dfrac{1}{\nu}(1 - \dfrac{1}{1+\widetilde{a}})\mathcal{L}\widetilde{u}, \\ (\widetilde{a}, \widetilde{u})|_{t=0} = (\widetilde{a}_0, \widetilde{u}_0), \end{cases}$$

where $(\widetilde{a}_0, \widetilde{u}_0) := (a_0/\bar\rho, u_0/c)$. Dropping the tildes on $(\widetilde{a}, \widetilde{u})$ the corresponding system for (a, v, w) now reads as follows:

$$\begin{cases} \partial_t a + \Lambda v = F, \\ \partial_t v - \Delta v - \Lambda a = \Lambda^{-1}\operatorname{div} G, \\ \partial_t w - \dfrac{\mu}{\nu}\Delta w = \Lambda^{-1}\operatorname{rot}\, G, \end{cases} \tag{1.8}$$

where

$$
(1.9)\qquad \begin{aligned} & F := -\mathrm{div}\,(au), \\ \text{and} \quad & G := -u\cdot\nabla u + (1 - \frac{P'(\bar\rho(a+1))}{c^2(a+1)})\nabla a - \frac{1}{\nu}(1 - \frac{1}{1+a})\mathcal{L}u. \end{aligned}
$$

The rest of the paper unfolds as follows: in the second section, we establish a linear estimate in the Fourier-Herz spaces for the linearized system of (1.1). In the third section, we prove Theorem 1.5. The last section is devoted for some technical lemmas used in the proof of Theorem 1.5. From hereon, C stands for harmless generic constants that may change from line to line.

§ 2. Linear estimate in the Fourier-Herz spaces

We consider the hyperbolic-parabolic system

$$
(2.1)\qquad \begin{cases} \partial_t a + \mathrm{div}\, u = f, \\ \partial_t u - \dfrac{1}{\nu}\mathcal{L}u + \nabla a = g, \\ (a,u)|_{t=0} = (a_0, u_0) \end{cases}
$$

and derive a linear a priori estimate in Fourier-Herz spaces using the method of Lyapunov in the frequency space. The following proposition states that we may get exactly the same estimates in the Fourier-Herz spaces as in the Besov spaces $\dot B^s_{2,1}(\mathbb{R}^d)$.

Proposition 2.1. *Let $s\in\mathbb{R}$, $1\le p\le\infty$ and (a,u) satisfy (2.1). For any $k_0\in\mathbb{Z}$, there exists some constant C depending only on k_0, λ and μ such that the following estimates hold :*

$$
\begin{aligned} &\|(a,u)\|^L_{\widetilde{L^\infty_t}(\dot{\mathcal{B}}^s_{p,\sigma})} + \|(\Lambda^2 a, \Lambda^2 u)\|^L_{L^1_t(\dot{\mathcal{B}}^s_{p,\sigma})} \\ &\qquad \le C\big(\|(a_0,u_0)\|^L_{\dot{\mathcal{B}}^s_{p,\sigma}} + \|(f,g)\|^L_{L^1_t(\dot{\mathcal{B}}^s_{p,\sigma})}\big) \end{aligned}
$$

and

$$
\begin{aligned} &\|(\Lambda a,u)\|^H_{\widetilde{L^\infty_t}(\dot{\mathcal{B}}^s_{p,\sigma})} + \|(\Lambda a, \Lambda^2 u)\|^H_{L^1_t(\dot{\mathcal{B}}^s_{p,\sigma})} \\ &\qquad \le C\big(\|(\Lambda a_0,u_0)\|^H_{\dot{\mathcal{B}}^s_{p,\sigma}} + \|(\Lambda f,g)\|^H_{L^1_t(\dot{\mathcal{B}}^s_{p,\sigma})}\big). \end{aligned}
$$

Remark. Taking $p=2$ yields the estimates obtained in [7].

Proof. The argument below is completely identical to that described in [7] but we give the details for readers' convenience. After projecting u onto the compressible part by $v = \Lambda^{-1}\mathrm{div}\, u$, we first treat the homogeneous system of the first two equations of System (1.8) :

$$\begin{cases} \partial_t a + \Lambda v = 0, \\ \partial_t v - \Delta v - \Lambda a = 0 \end{cases} \tag{2.2}$$

and later resort to Duhamel's principle to handle the outer forces. Taking the Fourier transform, System (2.2) is transformed into

$$\begin{cases} \partial_t \widehat{a} + |\xi|\widehat{v} = 0, \\ \partial_t \widehat{v} + |\xi|^2\widehat{v} - |\xi|\widehat{a} = 0. \end{cases}$$

We may write this as

$$\frac{d}{dt}\begin{bmatrix} \widehat{a} \\ \widehat{v} \end{bmatrix} = A(\xi)\begin{bmatrix} \widehat{a} \\ \widehat{v} \end{bmatrix} \quad \text{with} \quad A(\xi) := \begin{bmatrix} 0 & -|\xi| \\ |\xi| & -|\xi|^2 \end{bmatrix}. \tag{2.3}$$

The characteristic equation for the matrix $A(\xi)$ is $X^2 + |\xi|^2 X + |\xi|^2 = 0$, and $A(\xi)$ has two distinct eigenvalues given by:

$$\lambda_\pm(\xi) := -\frac{1}{2}|\xi|^2\left(1 \pm \sqrt{1 - \frac{4}{|\xi|^2}}\right).$$

From this, we readily gather the following informations:

1. In the low frequency ($|\xi| < 2$), $\lambda_\pm(\xi)$ are two complex conjugated eigenvalues

$$\lambda_\pm(\xi) := -\frac{1}{2}|\xi|^2\Big(1 \pm iS(|\xi|)\Big) \quad \text{with} \quad S(|\xi|) = \sqrt{\frac{4}{|\xi|^2} - 1}$$

   whose real part are $-|\xi|^2/2$. Therefore, in this regime, we expect to get a parabolic smoothing for both components.

2. On the other hand, in the high frequency ($|\xi| > 2$), the two real eigenvalues can be expressed as

$$\lambda_\pm(\xi) := -\frac{1}{2}|\xi|^2\Big(1 \pm iR(|\xi|)\Big) \quad \text{with} \quad R(|\xi|) = \sqrt{1 - \frac{4}{|\xi|^2}}$$

   whose asymptotic behaviors are as follows: $\lambda_+(\xi) \sim -|\xi|^2$ and $\lambda_-(\xi) \sim -1$ as $|\xi| \to \infty$. Therefore, a parabolic and a damped mode coexist.

According to the above ansatz, we seek to derive the natural regularities for $(\widehat{a}, \widehat{v})$. From (2.3), we immediately obtain the following sets of equalities:

$$\frac{1}{2}\frac{d}{dt}|\widehat{a}|^2 + |\xi|\Re(\widehat{v}|\widehat{a}) = 0, \tag{2.4}$$

$$\frac{1}{2}\frac{d}{dt}|\widehat{v}|^2+|\xi|^2|\widehat{v}|^2-|\xi|\Re(\widehat{v}|\widehat{a})=0, \tag{2.5}$$

and

$$\frac{d}{dt}\Re(\widehat{b}|\widehat{v})+|\xi||\widehat{v}|^2-|\xi||\widehat{a}|^2+|\xi|^2\Re(\widehat{v}|\widehat{a})=0, \tag{2.6}$$

where $\Re z$ denotes the real part of a complex number z and $(\cdot|\cdot)$ is the inner product in $\mathbb{C}$ (that is $(z_1|z_2):=z_1\,\bar{z}_2$ with $\bar{z}$ denoting the complex conjugate of z). The above implies

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}^2+|\xi|^2|(\widehat{a},\widehat{v})|^2=0 \quad \text{with} \quad \mathcal{L}^2:=2|(\widehat{a},\widehat{v})|^2+|\xi|^2|\widehat{a}|^2-2|\xi|^2\Re(\widehat{a}|\widehat{v}).$$

Young's inequality ensures that there exists a universal constant $C_0>0$ independent of $|\xi|$ such that

$$C_0^{-1}\mathcal{L}^2\leq|(\widehat{a},|\xi|\widehat{a},\widehat{v})|^2\leq C_0\mathcal{L}^2.$$

Therefore, we may confirm the exponential stability

$$\mathcal{L}^2(t)\leq e^{-c_0\min(1,|\xi|^2)t}\mathcal{L}^2(0),$$

for all $t\geq 0$ with some positive constant c_0. To handle the source terms $\widehat{f}$ and $\widehat{g}$ we combine the above linear estimates with Duhamel's principle to obtain

$$|(\widehat{a},|\xi|\widehat{a},\widehat{v})(t)|+\min(1,|\xi|^2)\int_0^t|(\widehat{a},|\xi|\widehat{a},\widehat{v})|d\tau\leq C\Big(|(\widehat{b},|\xi|\widehat{a},\widehat{v})(0)|+\int_0^t|(\widehat{f},|\xi|\widehat{f},\widehat{g})|d\tau\Big) \tag{2.7}$$

and

$$|(\widehat{b},\widehat{v})(t)|+\int_0^t|(\widehat{b},|\xi|^2\widehat{v})|d\tau\leq C(\nu)\Big(|(\widehat{b},\widehat{v})(0)|+\int_0^t|(\widehat{f},\widehat{g})|d\tau\Big) \quad \text{for} \quad \frac{1}{\nu}\sqrt{1+\frac{1}{\nu^2}}\leq|\xi|.$$

In order to track the parabolic behavior of v, it is only a matter of looking at v as the solution to the following ODE:

$$\partial_t\widehat{v}+|\xi|^2\widehat{v}=|\xi|\widehat{a}+\widehat{f}.$$

Performing the inner product with $\widehat{v}$, we get

$$\frac{1}{2}\frac{d}{dt}|\widehat{v}|^2+|\xi|^2|\widehat{v}|^2=\Re(|\xi|\widehat{a}+\widehat{f}\,|\,\widehat{v})\leq(|\xi||\widehat{a}|+|\widehat{f}|)|\widehat{v}|.$$

Hence

$$|\widehat{v}(t,\xi)|\leq e^{-\nu|\xi|^2t}|\widehat{v}(0)|+\int_0^t e^{-\nu|\xi|^2(t-\tau)}(|\xi||\widehat{a}|+|\widehat{f}|)(\tau,\xi)\,d\tau.$$

Integrating both sides of the above inequality from 0 to t and combining with (2.7), we obtain the full L^1-parabolic smoothing for v:

$$
\begin{aligned}
|(\widehat{a}, |\xi|\widehat{a}, \widehat{v})(t)| + \min(1, |\xi|^2) \int_0^t |\widehat{a}| d\tau + |\xi|^2 \int_0^t |\widehat{v}| d\tau \\
\leq C\Big(|(\widehat{a}, |\xi|\widehat{a}, \widehat{v})(0)| + \int_0^t |(\widehat{f}, |\xi|\widehat{f}, \widehat{g})| d\tau \Big)
\end{aligned}
$$

Granted with the above estimate in the frequency space, it is now straightforward to complete the proof of the proposition : it suffices to multiply the Littlewood-Paley decomposition $\widehat{\phi}_j$ and take $L^{p'}(\mathbb{R}^d)$ norms. Then taking the supremum on $[0,t]$ of both sides of the resulting inequality and summing up over $j \leq k_0$ or $j > k_0$ with a weight 2^{js} will lead to the estimates for the compressible part. As the incompressible component w satisfies a mere heat equation

$$
\partial_t w - \frac{\mu}{\nu} \Delta w = \Lambda^{-1} \mathrm{rot}\ g,
$$

combining with the parabolic estimate for w lead to the desired estimates. □

§ 3. Proof of Theorem 1.5

We denote by $\mathcal{P}$ and $\mathcal{P}^\perp$ the projections onto divergence-free and potential vector-fields, respectively. By decomposition $u = \mathcal{P}u + \mathcal{P}^\perp u$, $\mathcal{P}u$ is decoupled from a and $\mathcal{P}u^\perp$, and treated as a mere heat equation:

$$
\begin{cases}
\partial_t a + \operatorname{div} \mathcal{P}^\perp u = F, \\
\partial_t \mathcal{P}^\perp u - \Delta \mathcal{P}^\perp u - \nabla a = \mathcal{P}^\perp G, \\
\partial_t \mathcal{P}u - \dfrac{\mu}{\nu} \Delta \mathcal{P}u = \mathcal{P}G,
\end{cases}
\tag{3.1}
$$

where F and G are as in (1.9). Moreover, let us keep in mind that since the scalar quantity v is interrelated with the vector field $\mathcal{P}^\perp u$ through $\mathcal{P}^\perp u = -\Lambda^{-1} \nabla v$, bounding v or $\mathcal{P}^\perp u$ is equivalent in the homogeneous Besov framework. This justifies the use of Proposition 2.1 (which is for the scalar case) to bound $\mathcal{P}^\perp u$.

First step: estimates for the incompressible part of the velocity.

Recall that $\mathcal{P}u$ satisfies

$$
\partial_t \mathcal{P}u - \frac{\mu}{\nu} \Delta \mathcal{P}u = \mathcal{P}G.
$$

Therefore, splitting the above equation into low and high frequencies, and taking advantage of the endpoint maximal regularity estimate for the heat equation in both Besov spaces and Fourier-Herz spaces, we get

$$
\|\mathcal{P}u\|^L_{\widetilde{L^\infty_t}(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} + \frac{\mu}{\nu} \|\Lambda^2 \mathcal{P}u\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} \leq C\Big(\|\mathcal{P}u_0\|^L_{\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1}} + \|\mathcal{P}G\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} \Big)
\tag{3.2}
$$

and

$$\|\mathcal{P}u\|^H_{\widetilde{L_t^\infty}(\dot B^{\frac{d}{p}-1}_{p,1})} + \frac{\mu}{\nu}\|\Lambda^2\mathcal{P}u\|^H_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})} \leq C\Big(\|\mathcal{P}u_0\|^H_{\dot B^{\frac{d}{p}-1}_{p,1}} + \|\mathcal{P}G\|^H_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}\Big). \tag{3.3}$$

Second step: estimates for low frequencies.

Combining (3.2) and Proposition 2.1 with $s = \frac{d}{q} - 1$ and $1 \leq q \leq 2$ yields

$$\begin{aligned} &\|(a,u)\|^L_{\widetilde{L_t^\infty}(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} + \|\Lambda^2(a,u)\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} \\ &\qquad\qquad \leq C\big(\|(a_0,u_0)\|^L_{\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1}} + \|(F,G)\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})}\big). \end{aligned} \tag{3.4}$$

The above inequality holds true for any value of the threshold $k_0 \in \mathbb{Z}$, as it is for the low frequency (see Proposition 2.1). The constant C depends on k_0, but that dependency will be irrelevant for our purpose.

Third step: estimates for high frequency.

To handle high frequencies, we shall take advantage of the *effective velocity field* used by Haspot [8, 9] to study the standard barotropic Navier-Stokes equations in the low regularity Besov framework. See also Danchin's survey [7] for a concise exposition. A similar field (named *effective viscous flux*) has been used before by Hoff [10] in a slightly different context. Haspot made extensive use of that quantity to decouple the density and the velocity, as well as to avoid some technical *para-linearization* technique used in [3, 5] in order to control the loss of regularity in the high frequency scheme.

The basic idea is to introduce a new vector-field e so that $-\Delta\mathcal{P}^\perp u + \nabla a = -\Delta e$, that is to say

$$e := \mathcal{P}^\perp u + \nabla(-\Delta)^{-1}a = \nabla(-\Delta)^{-1}(a - \operatorname{div} u).$$

We rewrite the second equation of (3.1) using e into

$$\partial_t e - \Delta e = \mathcal{P}^\perp G + \nabla(-\Delta)^{-1}\partial_t a.$$

Since $\partial_t a = F - \operatorname{div} u = F - \operatorname{div} e + a$, we have in the end

$$\partial_t e - \Delta e = \nabla(-\Delta)^{-1}(F - \operatorname{div} G) + e - \nabla(-\Delta)^{-1}a.$$

The crucial observation is that the terms e and $\nabla(-\Delta)^{-1}a$ on the right hand-side are of *lower order* in high frequencies.

Now, resorting to Lemma 4.3 and using the fact that the Fourier multiplier $\nabla(-\Delta)^{-1}$ is homogeneous of degree -1, we get

$$\begin{aligned} \|e_H\|_{\widetilde{L_t^\infty}(\dot B^{\frac{d}{p}-1}_{p,1})} + \|\Lambda^2 e_H\|_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})} \leq C\Big(&\|e_{0H}\|_{\dot B^{\frac{d}{p}-1}_{p,1}} + \|(F - \operatorname{div} G)_H\|_{L^1_t(\dot B^{\frac{d}{p}-2}_{p,1})} \\ &+ \Big\|\Big(e - \nabla(-\Delta)^{-1}a\Big)_H\Big\|_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}\Big). \end{aligned} \tag{3.5}$$

Now, since $\operatorname{div} e = -a + \operatorname{div} u$, we may rewrite the first equation of (3.1) as

$$\partial_t a + \operatorname{div}(au) + a + \operatorname{div} e = 0,$$

which can be seen as a nonlinear transport equation with a damping term

$$\partial_t a + u \cdot \nabla a + a = -a \operatorname{div} u - \operatorname{div} e.$$

By spectral localization, we have

$$\partial_t a_k + u \cdot \nabla a_k + a_k = -\dot{\Delta}_k(a \operatorname{div} u) - \operatorname{div} e_k + [u \cdot \nabla, \dot{\Delta}_k]a,$$

where $a_k := \dot{\Delta}_k a$ and so on. Standard energy computation for the transport equation give us

$$\frac{d}{dt}\|a_k\|_{L^p} + \|a_k\|_{L^p} \le \frac{1}{p}\|\operatorname{div} u\|_{L^\infty}\|a_k\|_{L^p} + \|\dot{\Delta}_k(a \operatorname{div} u) + \operatorname{div} e_k + [u \cdot \nabla, \dot{\Delta}_k]a\|_{L^p}.$$

Integrating the above with respect to time, we obtain

$$\begin{aligned}(3.6)\quad \|a_k\|_{L^p} + \int_0^t \|a_k\|_{L^p} d\tau \le \|a_{0k}\|_{L^p} + \frac{1}{p}\int_0^t \|\operatorname{div} u\|_{L^\infty}\|a_k\|_{L^p} \\ + \int_0^t \big(\|\operatorname{div} e_k\|_{L^p} + \|\dot{\Delta}_k(a \operatorname{div} u)\|_{L^p} + \|[u \cdot \nabla, \dot{\Delta}_k]a\|_{L^p}\big) d\tau.\end{aligned}$$

By Lemma 4.8, we have for the commutator:

$$\|[u \cdot \nabla, \dot{\Delta}_j]a\|_{L^p} \le C c_k 2^{-k\frac{d}{p}} \|\nabla u\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \quad \text{with} \quad \sum_{k \in \mathbb{Z}} c_k = 1$$

and, by Lemma 4.5,

$$\|\dot{\Delta}_k(a \operatorname{div} u)\|_{L^p} \le C c_k 2^{-k\frac{d}{p}} \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|\operatorname{div} u\|_{\dot{B}^{\frac{d}{p}}_{p,1}}.$$

Therefore, multiplying (3.6) by $2^{k\frac{d}{p}}$ and summing up over $k \ge k_0$ yields

$$\begin{aligned}(3.7)\quad \|a\|^H_{\widetilde{L^\infty_t}(\dot{B}^{\frac{d}{p}}_{p,1})} + \|a\|^H_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} \\ \le C\Big(\|a_0\|^H_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|\operatorname{div} e\|^H_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \int_0^t \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|\operatorname{div} u\|_{\dot{B}^{\frac{d}{p}}_{p,1}} d\tau\Big).\end{aligned}$$

Note that in the above, we already have a control on $\|\operatorname{div} e_H\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}$ by (3.5). Using also (3.7), we thus obtain

$$\begin{aligned}\|a\|^H_{\widetilde{L^\infty_t}(\dot{B}^{\frac{d}{p}}_{p,1})} + \|a\|^H_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|e\|^H_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\Lambda^2 e\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \\ \le C\Big(\|a_0\|^H_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|u_0\|^H_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \int_0^t \Big(\|(F - \operatorname{div} G)\|^H_{\dot{B}^{\frac{d}{p}-2}_{p,1}} + \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|\operatorname{div} u\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big) d\tau \\ + \Big\|\Big(e - \nabla(-\Delta)^{-1}a\Big)\Big\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\Big).\end{aligned}$$

Let us observe that due to the restriction on the summation, we have

$$\|e\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \leq C2^{-2k_0}\|\Lambda^2 e\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}$$

and

$$\|\nabla(-\Delta)^{-1}a\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \leq C2^{-2k_0}\|a\|^H_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

Hence choosing k_0 large enough, we may absorb the last term of the above into left hand-side to obtain

$$\begin{aligned} \|a\|^H_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|a\|^H_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|e\|^H_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\Lambda^2 e_H\|_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \leq C\Big(\|a_0\|^H_{\dot{B}^{\frac{d}{p}}_{p,1}} \\ + \|u_0\|^H_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \int_0^t \Big(\|(F-\operatorname{div} G)\|^H_{\dot{B}^{\frac{d}{p}-2}_{p,1}} + \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\nabla u\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)d\tau\Big). \end{aligned} \tag{3.8}$$

We notice that $u = \mathcal{P}u + e - \nabla(-\Delta)^{-1}a$. Therefore,

$$\begin{aligned} \|u\|^H_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|u\|^H_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})} \\ \leq \|\mathcal{P}u\|^H_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\mathcal{P}u\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \\ + C\Big(\|e\|^H_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|e\|^H_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|a\|^H_{L^\infty_t(\dot{B}^{\frac{d}{p}-2}_{p,1})} + \|a\|^H_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big). \end{aligned} \tag{3.9}$$

Putting together (3.2), (3.3), (3.4), (3.8) and (3.9), we finally obtain

$$\begin{aligned} X(t) \leq C\Big(X_0 + \\ \int_0^t \Big(\|(F,G)\|^L_{\dot{\mathcal{B}}^{\frac{n}{q}-1}_{q,1}} + \|F\|^H_{\dot{B}^{\frac{d}{p}-2}_{p,1}} + \|G\|^H_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\nabla u\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)d\tau\Big) \end{aligned} \tag{3.10}$$

for some constant $C = C(d,p,\mu,\lambda,\kappa,P)$.

Last step: nonlinear estimates.

Since $2 \leq p$, Hausdorff-Young inequality impilies the continuous embedding $\dot{\mathcal{B}}^s_{p,1}(\mathbb{R}^d) \hookrightarrow \dot{B}^s_{p,1}(\mathbb{R}^d)$ for all $s \in \mathbb{R}$. This along with Sobolev embeddings ensures that Fourier-Herz regularities in the low frequencies imply the "natural" Besov regularities : we have $(a,u)_L \in \widetilde{L^\infty_t}(\dot{B}^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d)) \cap L^1(0,t;\dot{B}^{\frac{d}{p}+1}_{p,1}(\mathbb{R}^d))$. More precisely, one may simply note the following continuous embeddings

$$\|f\|_{\dot{B}^{s+\frac{d}{p}}_{p,1}} \leq C\|f\|_{\dot{\mathcal{B}}^{s+\frac{d}{p}}_{p,1}} \leq C\|f\|_{\dot{\mathcal{B}}^{s+\frac{d}{q}}_{q,1}} \tag{3.11}$$

for all $s \in \mathbb{R}$ and $q \leq 2 \leq p$.

We claim that the integral in the right hand-side of (3.10) may be bounded by $CX^2(t)$. Let us start with the bound for F. Since the product operates continuously on $\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$, we have

$$\|F\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-2}_{p,1})} \le \|F\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \le C\|au\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} \le C\|a\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|u\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})} \le CX^2(t).$$

Note that in the above, we used the fact that the summation is restricted in the high frequencies and also the embedding given in (3.11) to bound the low frequencies of a and u by $X(t)$. Next, as

$$G = -u\cdot\nabla u + (1 - \frac{P'(\bar\rho(a+1))}{c^2(a+1)})\nabla a - \frac{a}{1+a}\frac{1}{\nu}\mathcal{L}u,$$

it is easily seen that combining Lemma 4.5 and 4.6 leads to

$$\|G\|^H_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \le CX^2(t).$$

We omit the details here as the calculations are exactly the same to those when proving Lemma 1.3 (cf. [7]).

We next bound F and G at the low frequencies in $L^1(0,t;\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})$. We have

$$\|F\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{q}-1}_{q,1})} \le C\|au\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}}_{q,1})}.$$

Bony decomposition (see (4.4) in Appendix) gives us

$$au = \dot{T}_a u + \dot{R}(a,u) + \dot{T}_u a_L + \dot{T}_u a_H.$$

For the first three terms of the above, using Lemma 4.7 with $s = \frac{d}{p}+1$ and the fact that $a \in L^\infty(0,t;\dot{B}^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d)\cap\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d))\cap L^1(0,t;\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d))$, we infer that

$$\|\dot{T}_a u\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}}_{q,1})} \le C\|a\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|u\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})},$$

$$\|\dot{T}_u a_L\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}}_{q,1})} \le C\|u\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|a_L\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}$$

and

$$\|\dot{R}(a,u)\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}}_{q,1})} \le C\|a\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|u\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})},$$

under our restriction on q. The last term has to be treated differently since there is no smoothing for the high frequencies of a; a_H only belongs to $L^\infty(0,t;\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d))\cap L^1(0,t;\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d))$. However, due to the spectral cut-off and Lemma 4.7, we have

$$\|\dot{T}_u a_H\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}}_{q,1})} \le \|\dot{T}_u a_H\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} \le C\|u\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|a_H\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

Therefore, we obtain

$$\|F\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} \le CX^2(t).$$

Next, to bound the convective term of G_L, we use again Bony decomposition

$$u \cdot \nabla u^i = T_u \cdot \nabla u^i + \sum_{j=1}^{n} R(u^j, \partial_j u^i) + T_{\nabla u^i} \cdot u, \qquad i \in \{1, \cdots, n\}$$

and apply Lemma 4.7. This gives us

$$\|T_u \cdot \nabla u^i\|_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} \le C\|u\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \|\nabla u\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})},$$

$$\|T_{\nabla u^i} \cdot u\|_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} \le C\|\nabla u^i\|_{L^2_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \|u\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}$$

and

$$\|R(u^j, \partial_j u^i)\|_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} \le C\|u\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \|\nabla u\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

Therefore, we have $\|u \cdot \nabla u\|_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} \le CX^2(t)$.

To handle the second term of G_L, let us recall that

$$\left(1 - \frac{P'(\bar\rho(1+a))}{c^2(1+a)}\right)\nabla a = \frac{1}{P'(\bar\rho)}\left(P'(\bar\rho) - \frac{P'(\bar\rho(1+a))}{(1+a)}\right)\nabla a$$

and that $Q(a) := P'(\bar\rho) - \frac{P'(\bar\rho(1+a))}{(1+a)}$ belongs to the same Besov space as a by Lemma 4.6. Using the decomposition

$$Q(a)\nabla a = T_{Q(a)}\nabla a_L + T_{Q(a)}\nabla a_H + R(Q(a), \nabla a) + T_{\nabla a} Q(a),$$

Lemma 4.7 and Lemma 4.6 give us that

$$\begin{aligned} \|T_{\nabla a} Q(a)\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} + \|R(Q(a), \nabla a)\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{d}{q}-1}_{q,1})} &\le C\|\nabla a\|_{L^2_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \|Q(a)\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})} \\ &\le C\|a\|^2_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}. \end{aligned}$$

For the first term, noticing that $a \in L^\infty(0,t; \dot{B}^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d))$ thanks to the restriction on the summation, we may use Lemma 4.7 and Lemma 4.6 to obtain

$$\begin{aligned} \|T_{Q(a)}\nabla a_L\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{q}-1}_{q,1})} &\le C\|T_{Q(a)}\nabla a_L\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{q}-2}_{q,1})} \\ &\le C\|Q(a)\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \|\nabla a_L\|_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \\ &\le C\|Q(a)\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \|a_L\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} \\ &\le C\|a\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \|a_L\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}. \end{aligned}$$

For the last term, we have by the spectral cut-off and Lemma 4.7 again

$$\begin{aligned}\|T_{Q(a)}\nabla a_H\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{q}-1}_{q,1})} &\leq C\|T_{Q(a)}\nabla a_H\|^L_{L^1_t(\dot{\mathcal{B}}^{\frac{n}{q}-2}_{q,1})}\\ &\leq C\|Q(a)\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|\nabla a_H\|_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\\ &\leq C\|a\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|a_H\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}.\end{aligned}$$

To estimate the last term, namely $\mathcal{N}G(a)\mathcal{L}u$, we use the decomposition

$$G(a)\mathcal{L}u = T_{G(a)}\mathcal{L}u_L + T_{G(a)}\mathcal{L}u_H + R(G(a),\mathcal{L}u) + T_{\mathcal{L}u}G(a),$$

and proceed in exactly the same way as for $Q(a)\nabla a$.

Putting all the previous estimates together, we have proved the desired inequality:

$$X(t) \leq C(X_0 + X^2(t))$$

for $d \geq 2$ and p satisfying both $2 \leq p \leq \min\{2q, \frac{dq}{d-q}\}$ and $2 \leq p < 2d$.

§ 4. Appendix

For the reader convenience, we here recall some technical results without proof that were needed in the previous sections.

§ 4.1. Fourier-Herz spaces

This section is devoted to the study of Fourier-Herz spaces $\dot{\mathcal{B}}^s_{p,\sigma}(\mathbb{R}^d)$. Obvious relations with standard Besov spaces hold:

$$\|f\|_{\dot{\mathcal{B}}^s_{p,\sigma}} \leq \|f\|_{\dot{B}^s_{p,\sigma}} \text{ if } 2 \leq p \leq \infty, \text{ and } \|f\|_{\dot{B}^s_{p,\sigma}} \leq \|f\|_{\dot{\mathcal{B}}^s_{p,\sigma}} \text{ if } 1 \leq p \leq 2.$$

The Bernstein-type lemma in the Fourier-Herz space is as follows.

Proposition 4.1 (Bernstein-type lemma)**.** *Let $\mathcal{A}$ be an annulus and B a ball. A constant C exists such that for any nonnegative integer k, any couple (p,q) with $1 \leq q \leq p \leq \infty$, and any function u spectrally supported in the ball B or the annulus $\mathcal{A}$, we have*

$$\begin{aligned}&\|\widehat{D^k u}\|_{L^{p'}} \leq C^{k+1}\lambda^{k+d(\frac{1}{q}-\frac{1}{p})}\|\widehat{u}\|_{L^{q'}}, && \text{if} \quad \operatorname{supp}\widehat{u} \subset \lambda B,\\ &C^{-k-1}\lambda^k\|\widehat{u}\|_{L^{p'}} \leq \|\widehat{D^k u}\|_{L^{p'}} \leq C^{k+1}\lambda^k\|\widehat{u}\|_{L^{p'}} && \text{if} \quad \operatorname{supp}\widehat{u} \subset \lambda\mathcal{C},\end{aligned} \tag{4.1}$$

where p' and q' are Hölder conjugates.

Proof. Let u be supported in an annulus λB and ψ be a $C_0^\infty(\mathbb{R}^d)$-function such that

$$\psi(\xi) := \begin{cases} 1 & \text{if} \quad \xi \in B \\ 0 & \text{if} \quad \text{else.} \end{cases}$$

Since $\|\widehat{D^k u}\|_{L^p} = \||\xi|^k \widehat{u}\|_{L^p} \leq C\lambda^k \|\widehat{u}\|_{L^p}$, we may consider only the case $k = 0$ without losing generality. Denoting $\psi_\lambda(\xi) := \psi(\lambda^{-1}\xi)$, we have by Hölder's inequality

$$\|\widehat{u}\|_{L^p} = \|\psi_\lambda \widehat{u}\|_{L^p} \leq \|\psi_\lambda\|_{L^{\widetilde{q}}} \|\widehat{u}\|_{L^q}$$

with

$$\frac{1}{\widetilde{q}} = \frac{1}{p'} - \frac{1}{q'} = \frac{1}{q} - \frac{1}{p}, \quad 1 \leq p' \leq q' \leq \infty.$$

Since $\operatorname{supp} \psi \subset \lambda B$, we have

$$\|\psi_\lambda\|_{L^{\widetilde{q}}} = \left(\int_{\mathbb{R}^d} |\psi(\lambda^{-1}\xi)|^{\widetilde{q}} d\xi \right)^{\frac{1}{\widetilde{q}}} = \lambda^{\frac{d}{\widetilde{q}}} \|\psi\|_{L^{\widetilde{q}}},$$

which proves the first assertion of Proposition 4.1 with $k = 0$. The second assertion follows from a similar argument. We omit the detail here. $\square$

The above inequality immediately implies the Sobolev-type embedding : If $1 \leq q \leq p \leq \infty$, then

$$\dot{\mathcal{B}}^{s+\frac{d}{q}}_{q,1}(\mathbb{R}^d) \hookrightarrow \dot{\mathcal{B}}^{s+\frac{d}{p}}_{p,1}(\mathbb{R}^d).$$

In particular, if $q \leq 2 \leq p_1 \leq p_2 \leq p_3 \leq \infty$, then we have continuous embeddings

$$\begin{aligned} \dot{\mathcal{B}}^{s+\frac{d}{q}}_{q,1}(\mathbb{R}^d) \hookrightarrow B^{s+\frac{d}{2}}_{2,1}(\mathbb{R}^d) \hookrightarrow \dot{\mathcal{B}}^{s+\frac{d}{p_1}}_{p_1,1}(\mathbb{R}^d) \hookrightarrow \dot{\mathcal{B}}^{s+\frac{d}{p_2}}_{p_2,1}(\mathbb{R}^d) \\ \hookrightarrow \dot{B}^{s+\frac{d}{p_2}}_{p_2,1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s+\frac{d}{p_3}}_{p_3,1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s}_{\infty,1}(\mathbb{R}^d). \end{aligned}$$

The first three and the last two relations are due to monotonicity property of Sobolev embeddings (in Besov spaces and Fourier-Herz spaces) and the third one is due to Hausdorff-Young inequality, which holds for exponents greater than 2.

§ 4.2. Regularity estimates for the linear heat equation

Consider

$$\begin{cases} \partial_t u - \nu \Delta u = f, & t > 0, \quad x \in \mathbb{R}^n, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{4.2}$$

where $\nu > 0$ and u_0, f are given.

It is known that optimal regularity estimates in Besov spaces for Equation (4.2) have to be stated in terms of the norm (1.4).

Lemma 4.2 ([1, 7]). *Let $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ satisfy the heat equation* (4.2)*. Then for all $1 \leq p \leq \infty$, $s \in \mathbb{R}$, $1 \leq q_1 \leq q_2 \leq \infty$, we have :*

$$\|u\|_{\widetilde{L_t^{q_1}}(\dot{B}_{p,1}^{s+\frac{2}{q_1}})} \leq C\big(\|u_0\|_{\dot{B}^s_{p,1}} + \|f\|_{\widetilde{L_t^{q_2}}(\dot{B}_{p,1}^{s-2+\frac{2}{q_2}})}\big) \quad \text{for all} \quad t \geq 0.$$

§ 4.3. Regularity estimates for the linear heat equation in Fourier-Herz spaces

We also have the estimate for (4.2) in the Fourier-Herz spaces.

Lemma 4.3. *Let $T > 0$, $s \in \mathbb{R}$ and $1 \leq p, r, \sigma \leq \infty$. Assume that $u_0 \in \dot{\mathcal{B}}^s_{p,\sigma}$ and $f \in \widetilde{L^r_T}(\dot{\mathcal{B}}_{p,\sigma}^{s-2+\frac{2}{r}})$. Then* (4.2) *has a unique solution u in $\widetilde{L^\infty_T}(\dot{\mathcal{B}}^s_{p,\sigma}) \cap \widetilde{L^r_T}(\dot{\mathcal{B}}_{p,\sigma}^{s+\frac{2}{r}})$ and there exists a constant C depending only on q and r such that for all $q \in [r, \infty]$, we have*

$$\nu^{\frac{1}{q}}\|u\|_{\widetilde{L^q_T}(\dot{\mathcal{B}}_{p,\sigma}^{s+\frac{2}{q}})} \leq C\Big(\|u_0\|_{\dot{\mathcal{B}}^s_{p,\sigma}} + \nu^{\frac{1}{r}-1}\|f\|_{\widetilde{L^r_T}(\dot{\mathcal{B}}_{p,\sigma}^{s-2+\frac{2}{r}})}\Big).$$

If in addition σ is finite, then u belongs to $C([0, T); \dot{\mathcal{B}}^s_{p,\sigma})$.

Proof. We sketch the proof. Since u_0 and f are tempered distributions, the equation (4.2) admits a unique solution $u \in \mathcal{S}'((0, T) \times \mathbb{R}^d)$ satisfying the integral equation

$$\widehat{u}(t, \xi) = e^{-t\nu|\xi|^2}\widehat{u}_0(\xi) + \int_0^t e^{-(t-\tau)\nu|\xi|^2}\widehat{f}(\tau, \xi)d\tau \tag{4.3}$$

in the sense of tempered distribution.

To prove Lemma 4.3, we first notice the heat kernel estimate: For any $1 \leq q \leq p \leq \infty$, we have

$$\|e^{-t\nu|\xi|^2}\widehat{\phi}_j\widehat{u}\|_{L^{p'}} \leq C2^{jd(\frac{1}{q}-\frac{1}{p})}e^{-c\nu t2^{2j}}\|\widehat{\phi}_j\widehat{u}\|_{L^{q'}},$$

which is just a consequence of Hölder's inequality. Multiplying $\widehat{\phi}_j$ to (4.3) and taking the $L^{p'}(\mathbb{R}^d)$ norm, we have

$$\begin{aligned}\|\widehat{\phi}_j\widehat{u}(t)\|_{L^{p'}} &\leq \|e^{-t|\cdot|^2}\widehat{\phi}_j\widehat{u}_0(\cdot)\|_{L^{p'}} + \int_0^t \|e^{-(t-\tau)|\cdot|^2}\widehat{\phi}_j\widehat{f}(\tau, \cdot)\|_{L^{p'}}d\tau \\ &\leq e^{-c\nu t2^{2j}}\|\widehat{\phi}_j\widehat{u}_0(\cdot)\|_{L^{p'}} + \int_0^t e^{-c\nu(t-\tau)2^{2j}}\|\widehat{\phi}_j\widehat{f}(\tau, \cdot)\|_{L^{p'}}d\tau\end{aligned}$$

Now, it is a matter of using the convolution inequality as in the proof of Lemma 4.2 and taking the weighted summation to make the Fourier-Herz norms. ☐

§ 4.4. Estimates for product, composition and commutators

For any couple (u, v) of tempered distributions, we have the following (formal) decomposition of uv:

$$
\begin{aligned}
uv &= \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\ \dot{\Delta}_j v + \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}v\ \dot{\Delta}_j u + \sum_{j\in\mathbb{Z}} \sum_{|k-j|\le 1} \dot{\Delta}_j u\, \dot{\Delta}_k v \\
&=: T_u v + T_v u + R(u, v).
\end{aligned}
\tag{4.4}
$$

Clearly, the first two terms are defined for any couple (u, v) in $\mathcal{S}(\mathbb{R}^d)'$ as the series is locally finite in the Fourier space. As for the last so-called remainder term, it is also defined if, roughly speaking, the sum of the regularity indices of u and of v is positive. This is detailed in the following lemma, the proof of which may be found in e.g. [1, 7].

Lemma 4.4. *Let $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$ and $t < 0$. We have*

$$\|T_u v\|_{\dot{B}^s_{p,r}} \le C\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} \quad \text{and} \quad \|T_u v\|_{\dot{B}^{s+t}_{p,r}} \le C\|u\|_{\dot{B}^t_{\infty,\infty}}\|v\|_{\dot{B}^s_{p,r}}.$$

Let $(s_j, p_j, r_j) \in \mathbb{R} \times [1, \infty]^2$ for $j = 1, 2$. We have

- *if $s_1 + s_2 > 0$, $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \le 1$ and $\frac{1}{r} := \frac{1}{r_1} + \frac{1}{r_2} \le 1$ then*

$$\|R(u, v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \le C\|u\|_{\dot{B}^{s_1}_{p_1,r_1}}\|v\|_{\dot{B}^{s_2}_{p_2,r_2}};$$

- *if $s_1 + s_2 = 0$, $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \le 1$ and $\frac{1}{r_1} + \frac{1}{r_2} \ge 1$ then*

$$\|R(u, v)\|_{\dot{B}^0_{p,\infty}} \le C\|u\|_{\dot{B}^{s_1}_{p_1,r_1}}\|v\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

As a corollary of the above lemma, we have the following general product estimate.

Lemma 4.5 ([1, 7]). *Let $\delta \ge 0$ and $-\min\left(\frac{n}{p}, \frac{n}{p'}\right) < \sigma \le \frac{n}{p} - \delta$. Then we have*

$$\|uv\|_{\dot{B}^\sigma_{p,1}} \le C\|u\|_{\dot{B}^{\frac{n}{p}-\delta}_{p,1}}\|v\|_{\dot{B}^{\sigma+\delta}_{p,1}}.$$

Lemma 4.6 ([4]). *Let I be an open interval of $\mathbb{R}$ containing 0, and $F : I \to \mathbb{R}$, a smooth function vanishing at 0. Then for any $s > 0$, $1 \le p \le \infty$ and interval J compactly supported in I there exists a constant C such that*

$$\|F(a)\|_{\dot{B}^s_{p,1}} \le C\|a\|_{\dot{B}^s_{p,1}} \quad \text{for any} \quad a \in \dot{B}^s_{p,1} \quad \text{valued in} \quad J.$$

In the case $s > -\min(n/p, n/p')$ if in addition to the above hypotheses we have $a \in \dot{B}^{\frac{n}{p}}_{p,1}$ then $F(a) \in \dot{B}^{\frac{n}{p}}_{p,1} \cap \dot{B}^s_{p,1}$ and

$$\|F(a)\|_{\dot{B}^s_{p,1}} \le \|a\|_{\dot{B}^s_{p,1}}\left(|F'(0)| + C\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right).$$

We next investigate the boundedness properties of paraproduct operators from Besov spaces to Fourier-Herz spaces. It is a trivial generalization of corresponding "p to 2" estimate in [7] (indeed, taking $q = 2$ below recovers the estimate in [7]).

Lemma 4.7. *Let $d \geq 2$ and $s \in \mathbb{R}$, $1 \leq q \leq 2$ and $2 \leq p \leq \min(2q, \frac{qd}{d-q})$. Then we have*

$$\|\dot{T}_a b\|_{\dot{\mathcal{B}}^{s-1+\frac{d}{q}-\frac{d}{p}}_{q,1}} \leq C\|a\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|b\|_{\dot{B}^s_{p,1}}.$$

Let $d \geq 2$, $s > 1 - \min(\frac{d}{p}, \frac{d}{p'})$, $1 \leq q \leq 2$ and $2 \leq p \leq 2q$. Then we have

$$\|\dot{R}(a,b)\|_{\dot{\mathcal{B}}^{s-1+\frac{d}{q}-\frac{d}{p}}_{q,1}} \leq C\|a\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|b\|_{\dot{B}^s_{p,1}}.$$

Proof. The proof is similar to the one that appears in [7].

To prove the estimate for the off-diagonal part, we consider two cases separately : $p \leq d$ and $p \geq d$. Let us recall the property of spectral cut-off

$$\dot{\Delta}_j(\dot{T}_a u) = \sum_{|k-j|\leq 3} \dot{\Delta}_j\big(\dot{S}_{k-3}a\dot{\Delta}_k u\big).$$

We first consider the case $p \leq d$. We have a canonical embedding $\dot{B}^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d)$ in this case. Since $2 \leq q' \leq \infty$, we have

$$\begin{aligned}\|\widehat{\phi}_j\widehat{\dot{T}_a b}\|_{L^{q'}} &\leq \sum_{|k-j|\leq N_0} \|\widehat{\phi}_j\big(\widehat{\dot{S}_{k-3}a} * (\widehat{\phi}_k\widehat{b})\big)\|_{L^{q'}}\\ &\leq \sum_{|k-j|\leq N_0} \|\dot{\Delta}_j\big(\dot{S}_{k-3}a\dot{\Delta}_k b\big)\|_{L^q}.\end{aligned}$$

Taking $\widetilde{q}$ such that $q \geq \widetilde{q} \geq 1$ and $\frac{1}{\widetilde{q}} = \frac{1}{d} + \frac{1}{p}$ with $\widetilde{q} \leq d$ and $\widetilde{q} \leq p$, Hölder's inequality gives

$$\begin{aligned}\|\widehat{\phi}_j\widehat{\dot{T}_a b}\|_{L^{q'}} \leq &C \sum_{|k-j|\leq N_0} 2^{jd(\frac{1}{\widetilde{q}}-\frac{1}{q})}\|\dot{\Delta}_j\big(\dot{S}_{k-3}a\dot{\Delta}_k b\big)\|_{L^{\widetilde{q}}}.\\ \leq &C2^{j(\frac{d}{p}-\frac{d}{q}+1)} \sum_{|k-j|\leq N_0} \|\dot{S}_{k-3}a\|_{L^d}\|\dot{\Delta}_k b\|_{L^p}\\ \leq &C\|a\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} 2^{j(\frac{d}{p}-\frac{d}{q}+1)} \sum_{|j-k|\leq N_0} \|\dot{\Delta}_k b\|_{L^p}.\end{aligned}$$

Note that $1 \leq \widetilde{q} \leq q$ and $\frac{1}{\widetilde{q}} = \frac{1}{d} + \frac{1}{p}$ implies $\frac{d}{d-1} \leq p \leq \min\{d, \frac{dq}{d-q}\}$. From this, we also need $d \geq 2$.

We next consider the case $p \geq d$ (that is $d/p - 1 \leq 0$). We have if $1 \leq p/2 \leq q$

$$\begin{aligned}
\|\widehat{\phi}_j \widehat{\dot{T}_a u}\|_{L^{q'}} &\leq \sum_{|k-j|\leq N_0} \|\widehat{\phi}_j \big(\widehat{\dot{S}_{k-3} a} * (\widehat{\phi}_k \widehat{b})\big)\|_{L^{q'}} \\
&\leq \sum_{|k-j|\leq N_0} \|\dot{\Delta}_j \big(\dot{S}_{k-3} a \dot{\Delta}_k b\big)\|_{L^q} \\
&\leq C \sum_{|k-j|\leq N_0} 2^{jd(\frac{2}{p}-\frac{1}{q})} \|\dot{\Delta}_j \big(\dot{S}_{k-3} a \dot{\Delta}_k b\big)\|_{L^{\frac{p}{2}}} \\
&\leq C \sum_{|k-j|\leq N_0} 2^{jd(\frac{2}{p}-\frac{1}{q})} \|\dot{S}_{k-3} a\|_{L^p} \|\dot{\Delta}_k b\|_{L^p}.
\end{aligned}$$

After noticing that by $p \geq d$ we have

$$\|\dot{S}_{k-3} a\|_{L^p} \leq C 2^{-k(\frac{d}{p}-1)} \|a\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}},$$

we conclude the result for $2 \leq p \leq 2q$.

The estimate for the diagonal part is merely a consequence of Sobolev embedding. We have for $\dot{R}(u,v)$,

$$\dot{\Delta}_j(\dot{R}(u,v)) = \sum_{k>j-5} \dot{\Delta}_j(\dot{\Delta}_k u \, \widetilde{\dot{\Delta}}_k v),$$

where $\widetilde{\dot{\Delta}}_k v = \sum_{|l-k|\leq 2} \dot{\Delta}_l v$. Hence,

$$\|\widehat{\phi}_j \widehat{\dot{R}(u,v)}\|_{L^{q'}} \leq \sum_{k>j-5} \|\dot{\Delta}_j(\dot{\Delta}_k u \, \widetilde{\dot{\Delta}}_k v)\|_{L^q}.$$

Lemma 4.4 gives that if $\frac{1}{q} \leq \frac{2}{p}$ and $s_1 + s_2 + d\inf(0, 1-\frac{2}{p}) > 0$, then

$$\|\dot{R}(a,b)\|_{\dot{B}^{s_1+s_2+\frac{d}{q}-\frac{2d}{p}}_{q,1}} \leq C\|a\|_{\dot{B}^{s_1}_{p,1}} \|b\|_{\dot{B}^{s_2}_{p,1}}.$$

Taking $s_2 = s$ and $s_1 = d/p - 1$ yields the result. □

The following commutator estimates are classical (see e.g. [1] and the references therein).

Lemma 4.8 ([1, 7]). *Let $1 \leq p \leq \infty$ and $-\min\left(\frac{n}{p}, \frac{n}{p'}\right) < s \leq 1 + \frac{n}{p}\cdot$ Then we have*

$$\sum_{j\in\mathbb{Z}} 2^{js} \|[u\cdot\nabla, \dot{\Delta}_j]a\|_{L^p} \leq C\|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|a\|_{\dot{B}^s_{p,1}},$$

$$\sum_{j\in\mathbb{Z}} 2^{j(s-1)} \|[u\cdot\nabla, \partial_i\dot{\Delta}_j]a\|_{L^p} \leq C\|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|\nabla a\|_{\dot{B}^{s-1}_{p,1}}, \qquad i = 1, \cdots, n.$$

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